Factorials

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• The exclamation point has a special meaning in math. Basically you start at a number and multiply down until you get to 1. For a couple examples, $3! = 3 \cdot 2 \cdot 1 = 6$, and $4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$. Now simplify the following.

1. $5!$
2. $2!$
3. $8!$
4. $6!$

• For how to say it, 5! is "five factorial." Since 5!=120, you could say, "five factorial is one hundred twenty." You can also combine factorials with other arithmetic operations. For example, (2+3)!=5!=120 (remember to do the stuff in the parentheses first). Or 2!+3!=2+6=8. Now simplify the following.

1. $(4+1)!$
2. $2 \cdot 2!$
3. $(3!)^2$
4. $6! - 5!$
5. $(7-4)!$
6. $(4+1)! - 10$
7. $8!/8$
8. $(2!)^{3!}$

• Notice you didn't divide any big factorials in the above exercises. For example, suppose you want to find $40! \div 38!$. The problem with this is that 40! is a huge number, and so is 38!, so if you found both numbers then tried to divide them it would be dreadful. Fortunately, there's a shortcut. Observe $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$. You can think of peeling the 5 of the front and grouping the rest, or $5 \cdot (4 \cdot 3 \cdot 2 \cdot 1)$. And what's in the parentheses? 4! is. Long story short: $5! = 5 \cdot 4!$. Likewise, $7! = 7 \cdot 6!$, $10! = 10 \cdot 9!$, and so on.

You can even do it twice, so $13! = 13 \cdot 12 \cdot 11!$. And $8! = 8 \cdot 7 \cdot 6!$. Back to $40! \div 38!$...

$$\frac{40!}{38!} = \frac{40 \cdot 39 \cdot 38!}{38!} = \frac{40 \cdot 39 \cdot \cancel{38!}}{\cancel{38!}} = 40 \cdot 39 = 1560$$

Now you try it.

1. $10! \div 9!$
2. $\dfrac{11!}{9!}$
3. $21!/19!$
4. $8! \div 5!$
5. $(2+6)! \div 6!$
6. $\dfrac{9! \cdot 2}{7! \cdot 3}$
7. $100!/97!$
8. $3!/5!$

• Other than being cool, what's good about factorials? They tell you in how many ways you can line distinct things up. For example, if you have 5 Hot Wheels (a red, a blue, a purple, a black, and a white), in how many ways can you line them up? The answer is 5!, or 120, ways. By the way, the number of ways you can line a collection of objects up is called the number of **permuations**. So there are 120 permutations of 5 objects, or 120 ways to permute 5 objects.

Or suppose you have 8 people on a basketball team, and want them to line up for a photo. How many ways can they line up? 8!, or 40,320. In other words, there are 40,320 permutations of 8 objects, or 40,320 ways to permute 8 objects.

Now find the number of ways you can line up the following number of distinct objects.

1. 4
2. 10
3. 7
4. 13

• Suppose instead you have 6 different Hot Wheels and only want to line up 4 of them (so you want to line up 4 objects out of 6 total). This is like a factorial, but instead of multiplying from 6 all the way down to 1 (6 numbers total), you just do the first 4. So $6 \cdot 5 \cdot 4 \cdot 3$, or 360 ways. You can write this as $6! \div 2!$, since you aren't lining up 2 of them. Now find the number ways you can line up...

1. 3 objects out of 5 total
2. 2 objects out of 8 total
3. 6 objects out of 8 total
4. 4 objects out of 10 total
5. 5 objects out of 20 total
6. 3 objects out of 90 total

• Pretty much the last thing of the worksheet is this: what if you have 10 objects and want to take 6 of them but don't care about order? That is, instead of lining them up you just throw them in a pile. Or you have 10 friends and invite 6 of them to a movie: it doesn't matter the order they're invited in. Or you have 5 toppings you can choose from for a pizza, but only want 3 toppings.

Once you know how to do these *with* order, it's simple to do it *without* order. Just divide the number of permutations by the factorial of how many objects you're taking. So if you have 8 objects and want to know in how many ways you can take 4 of them, *with* order it's $8 \cdot 7 \cdot 6 \cdot 5$, and *without* order it's $(8 \cdot 7 \cdot 6 \cdot 5) \div 4! = 70$. Now find the number of ways you can choose *without* order...

1. 3 objects out of 5 total
2. 2 objects out of 8 total
3. 6 objects out of 8 total
4. 4 objects out of 10 total
5. 5 objects out of 20 total
6. 3 objects out of 90 total

By the way, the number of ways you can select things *without* order is called the number of **combinations**. Order $\rightarrow$ permutations. No order $\rightarrow$ combinations.

• For a final note, some calculators even have a factorial button. For example, if you want to compute 4! on a TI-83 Plus, first hit $\boxed{4}$, then hit $\boxed{\text{MATH}}$, then hit right until you get to PRB at the top, then go down to ! and hit $\boxed{\text{ENTER}}$. Your screen should display 4!. Hit $\boxed{\text{ENTER}}$ one last time and 24 should pop up. Viola.

You can also use nPr and nCr found under PRB to find numbers of permutations and combinations. For example, type 6nPr3 and hit $\boxed{\text{ENTER}}$. This gives you the number of ways you can line up 3 objects from 6 total, or $6 \cdot 5 \cdot 4$, or 120. Same for nCr and combinations. Try it! Check the above exercises, like for the first one type 5nCr3.

Lastly, for some more notation, the number of combinations of n objects taken r at a time is $\binom{n}{r}$. So if you have 10 friends and want to take 3 of them to the movie, the number of ways you can do that is $\binom{10}{3} = (10 \cdot 9 \cdot 8)/3! = 120$.

Oh, and if you were curious, 0!=1 and $\left(\frac{1}{2}\right)! = \frac{\sqrt{\pi}}{2}$. Yeah, weird.